OPTIMIZING THE TRACKING PROCESS UNDER RANDOM PERTURBATIONS

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We consider the optimal control of the tracking process in a system described either by linear differential equations or by recurrent algebraic equations. In Sects. 1 and 2 we set out the basic equations of tracking theory. The optimal tracking problem is formulated in Sect. 3, Sect. 4 contains some solutions of the basic equation. Sections 5 and 6 concern sample continuous and discrete tracking problems. Various aspects of tracking theory have been investigated elsewhere (see [1–3]). For example, basic equation (2.5) was derived in [1]. Our formulation of the tracking problem follows [4]. We assume, however, that random perturbations affect not only the measuring device, but also the motion of the object.

1. The discrete-time process. Let the state of a system be described by the n-dimensional phase coordinate vector $x(t)$, and let the time t assume discrete values $t_0, t_1, \ldots, t_N = T$, where $t_k < t_{k+1}$ for all $k = 0, 1, \ldots, N-1$. The state of the system is measured (tracked) at the instants t_k; the results of these measurements are the l_k-dimensional vectors $y(t_k)$. We assume that the motion (variation of state) of the system and the measurement process are specified in the form of the linear relations

$$x(t_{k+1}) = A_k x(t_k) + b_k + F_k \xi(t_k) \quad (k=0, 1, \ldots, N-1) \tag{1.1}$$

$$y(t_k) = Q_k x(t_k) + \eta(t_k) \quad (k=0, 1, \ldots, N) \tag{1.2}$$

Here $\xi(t_k)$ and $\eta(t_k)$ are random vectorial quantities of dimensions m_k and l_k, respectively. These quantities characterize the perturbations acting on the object and the measurement errors. We assume that the random quantities $\xi(t_k)$ and $\eta(t_s)$ are independent of each other for all $k = 0, 1, \ldots, N-1$ and $s = 0, 1, \ldots, N$. We also assume that the quantities $\xi(t_k)$, $\eta(t_k)$ have zero mathematical expectations and that their correlation matrices G_k (of dimensions $m_k \times m_k$) and B_k (of dimensions $l_k \times l_k$) are known. The term "correlation matrices" refers throughout the present paper to unnormalized correlation matrices (second-moment matrices).

Our assumption that the mathematical expectation of $\xi(t_k)$ is zero does not reduce generality, since this mathematical expectation can always be included in the vector b_k; the equality to zero of the mathematical expectation of $\eta(t_k)$ is equivalent to the absence of a systematic measurement error. The $n \times n$ matrix A_k, the $n \times m_k$ matrix F_k, the $l_k \times n$ matrix Q_k, and the n-dimensional vector b_k appearing in Eqs. (1.1), (1.2) are assumed to be given. The matrix Q_k characterizes the composition of the measurements made at the instant t_k.

Let the stochastic distribution of the vector $x(t_0 - 0)$ at the instant $t_0 - 0$ directly preceding the start of the process be known. We assume that this distribution is also normal, having the mathematical expectation x_0 and the correlation matrix D_0. The purpose of tracking is to be able to indicate the mathematical expectation and the correlation matrix of the phase-coordinate vector at any instant. These quantities (the mathematical expectation and the correlation matrix) vary first by virtue of equations of motion

(1.1) and second as a result of the measurements. We assume that all of the stochastic distributions are normal and that the measurement results are treated by the maximum-plausibility method [6].

We denote the mathematical expectation and correlation matrix for the vector $x\,(t_k - 0)$ (i.e. directly prior to the kth measurement) by x_k and D_k, and the same quantities at the instant $t_k + 0$ (i.e. directly after the kth measurement) by x_k^* and D_k^*. Using the maximum-plausibility method, we can show that the relations

$$x_k^* = x_k + D_k^* Q_k' B_k^{-1} [y\,(t_k) - Q_k x_k], D_k^* = (D_k^{-1} + Q_k' B_k^{-1} Q_k)^{-1}$$
$$(k = 0, 1, \ldots, N) \tag{1.3}$$

are valid. Here the primes denote transposes, and the exponents -1 inverse matrices. Formulas (1.3) are derived in [4]. There are no measurements between the instants t_k and t_{k+1}; the variation of the vector $x\,(t_k)$ during this interval is described by Eq.(1.1). Computing the mathematical expectation and the correlation matrix of both sides of linear equation (1.1), we obtain

$$x_{k+1} = A_k x_k^* + b_k, D_{k+1} = A_k D_k^* A_k' + F_k G_k F_k' \; (k = 0, 1, \ldots, N-1) \tag{1.4}$$

Recursion relations (1.3),(1.4) describe the variation of the mathematical expectation and dispersion of the vector $x\,(t)$ as a result of the tracking process and motion of the object. In order to carry out computations by means of these formulas we must specify the matrices and the vectors A_k, b_k, F_k occurring in equations of motion (1.1), the matrices Q_k characterizing the composition of measurements, the random perturbation correlation matrices B_k and G_k, the initial values x_0 and D_0, as well as the results of measuring $y\,(t_k)$.

2. The continuous-time process. We can consider the case of continuous time by taking the limits in the relations of Sect. 1 (*). Let us set

$$\tau = \frac{T - t_0}{N}, \quad t_k = t_0 + k\tau \quad (k = 0, 1, \ldots, N) \tag{2.1}$$

and introduce matrices $A\,(t)$, $F\,(t)$, $Q\,(t)$, $B\,(t)$, $G\,(t)$, $D\,(t)$ and the vectors $b\,(t)$, $z\,(t)$ such that

$$A_k = E + \tau A\,(t_k), \quad b_k = \tau b\,(t_k), \quad F_k = \tau F\,(t_k), \quad Q_k = Q\,(t_k)$$
$$B_k = \tau^{-1} B\,(t_k), \quad G_k = \tau^{-1} G\,(t_k), \quad D_k = D\,(t_k), \quad x_k = z\,(t_k) \tag{2.2}$$

Here E is an $n \times n$ identity matrix.

Let us substitute relations (2.1),(2.2) into Eqs. (1.1),(1.2) and take the limits as $\tau \to 0$, $N \to \infty$ for $N\tau = T - t_0 = \text{const}$. This yields

$$dx\,(t) = [A\,(t)\,x\,(t) + b\,(t) + F\,(t)\,\xi\,(t)]\,dt, \quad y\,(t) = Q\,(t)\,x\,(t) + \eta\,(t) \tag{2.3}$$

Let us substitute Eqs. (2.2) into (1.3), expand them in powers of τ, and omit small quantities of order higher than the first,

$$x_k^* = z\,(t_k) + \tau D\,(t_k)\,Q'\,(t_k)\,B^{-1}\,(t_k)\,[y\,(t_k) - Q\,(t_k)\,z\,(t_k)]$$

$$D_k^* = \{D^{-1}\,(t_k)\,[E + \tau D\,(t_k)\,Q'\,(t_k)\,B^{-1}\,(t_k)\,Q\,(t_k)]\}^{-1} =$$
$$= D\,(t_k) - \tau D\,(t_k)\,Q'\,(t_k)\,B^{-1}\,(t_k)\,Q\,(t_k)\,D\,(t_k)$$

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*) A stricter derivation is possible on the basis of random process theory (see [1, 2]).

Next, we substitute these relations as well as Eqs. (2.2) into formulas (1.4) and once again omit higher-order small terms,

$$z(t_{k+1}) = z(t_k) + \tau \{A(t_k) z(t_k) + b(t_k) + D(t_k) Q'(t_k) B^{-1}(t_k) \times \times [y(t_k) - Q(t_k) z(t_k)]\}$$

$$D(t_{k+1}) = D(t_k) + \tau [A(t_k) D(t_k) + D(t_k) A'(t_k) - - D(t_k) Q'(t_k) B^{-1}(t_k) Q(t_k) D(t_k) + F(t_k) G(t_k) F'(t_k)]$$

Taking the limit as $\tau \to 0$, we obtain the differential equations [1]

$$dz = [Az + b + DQ'B^{-1}(y - Qz)]\, dt \tag{2.4}$$

$$dD / dt = AD + DA' - DQ'B^{-1}QD + FGF' \tag{2.5}$$

Here we have omitted for simplicity the explicit dependence of all functions on t. The initial conditions for Eqs. (2.4), (2.5) are of the form $z(t_0) = x_0$, $D(t_0) = D_0$ and specify the mathematical expectation and correlation matrix for the phase vector $x(t_0)$ at the start of the process.

Both recursion relations (1.3), (1.4) for discrete-time process (1.1), (1.2) and differential equations (2.4), (2.5) for continuous-time process (2.3) describe the evolution of the mathematical expectation and correlation matrix for the phase vector. We note that in contrast to the corresponding equations for the mathematical expectation, Eqs. (1.3), (1.4), (2.5) for the correlation matrix are independent, first, of the vectors b_k or $b(t)$, and, second, of the measurements of $y(t)$. Equations (1.3), (1.4), (2.5) for the correlation matrix are therefore ordinary (nonstochastic) equations and can be solved prior to the execution of tracking operations.

It is advisable to consider differential equations (2.4), (2.5) instead of discrete equations (1.3), (1.4) if the measurements are carried out either continuously or discretely, but with sufficient frequency. The function $B(t)$ characterizes the measurement error per unit time. Equations (2.4), (2.5) are also applicable if the process is described by differential equation (2.3), and if the measurements are made at discrete instants. The function $B^{-1}(t)$ can then be expressed as a sum of delta functions of time.

Further on we shall use Eq. (2.5) to consider the optimization of the tracking problem for a system described by differential equation (2.3). We assume that the matrices $A(t)$, $F(t)$, $Q(t)$, $B(t)$, $G(t)$ occurring in the right side of Eq. (2.5) are known. The dimensions of these matrices are, respectively, $n \times n$, $n \times m$, $l \times n$, $l \times l$, $m \times m$. Here $m(t)$ is the dimension of the random perturbation vector ξ and $l(t)$ is the dimension of the vector y, i.e. the number of scalar parameters measured at the instant t. These dimensions can vary during motion.

3. The optimal tracking problem. Let us introduce the notation

$$V(t) = Q'(t) B^{-1}(t) Q(t), \qquad K(t) = F(t) G(t) F'(t) \tag{3.1}$$

and rewrite Eq. (2.5) in the form

$$dD / dt = AD + DA' - DVD + K, \qquad D(t_0) = D_0 \tag{3.2}$$

Relations (3.1) imply that the matrices $V(t)$ and $K(t)$ have the dimensions $n \times n$ and that they are symmetric and positive-definite, since these properties are also exhibited by the matrices B and G. The matrix V characterizes the tracking process and depends on the composition of the measurements (i.e. on the matrix Q) and on their exactness (i.e. on the matrix B). In particular, if no measurements are made then $V = 0$. The matrix K characterizes the perturbations acting on the object.

If the tracker is able to vary the set of measured parameters or the accuracy of measurements, then the matrix V in matrix equation (3.2) can be considered as a controlling function. The role of the phase coordinates in system (3.2) is played by the elements of the correlation matrix D. By virtue of the symmetry of the matrix D, the number of distinct elements is $n(n+1)/2$.

The controlling function V can be subjected to the restrictions

$$V(t) \in U(t), \qquad t_0 \leqslant t \leqslant T \tag{3.3}$$

where $U(t)$ is the closed set of matrices characterizing the tracking possibilities. Let us introduce the functionals

$$J_0 = \int_{t_0}^{T} f(V, t)\, dt, \qquad J = \sum_{j,\, k=1}^{n} D_{jk}(T_*)\, q_j q_k \tag{3.4}$$

Here the scalar function f is defined for all $t \in [t_0, T]$ and $V(t) \in U(t)$. By $D_{jk}(T_*)$ we denote the elements of the matrix D at the fixed instant T_* ($t_0 < T_* \leqslant T$), and by q_j the components of the given nonzero n-dimensional vector q. The functional J_0 in (3.4) characterizes the cost or duration of the tracking process; the function f is the cost of tracking per unit time.

For example, let the tracker have the option at any instant either of making the measurements in a set way (with the matrix V_0) or of making no measurements whatever. The set U for any t then consists of the two matrices 0 and V_0. If we set

$$f(0, t) = 0, \qquad f(V_0, t) = 1$$

in (3.4), then the functional J_0 is equal to the duration of observation.

It is clear that the functional J of (3.4) is equal to the dispersion of the following linear function of the phase coordinates at the instant T_*:

$$Z = (q, x(T_*)) \tag{3.5}$$

The parentheses denote the scalar product of the vectors.

We can now pose various problems of tracking optimization as optimal control problems for system (3.2). The system of equations, the initial conditions, and the restrictions on the controlling functions are specified by Eqs. (3.2), (3.3). It is natural to specify the functional to be minimized either as the integral functional J_0 (the cost or duration of tracking) or in the form of the functional J from (3.4); this is equivalent to minimizing the dispersion of the quantity (3.5). In the former case we can impose conditions of the form $J_i \leqslant c_i$, where the J_i are functionals of the type J from (3.4) and $c_i > 0$ are given numbers $i = 1, \ldots, s$. These conditions mean that the accuracy of determining certain parameters of the form (3.5) must fall within the specified range. In the latter case we can impose conditions of the form $J_i \leqslant c_i$, $i = 1, \ldots, s$, as well as a restriction on the cost or duration of the observations of the form $J_0 \leqslant c_0$, where $c_0 > 0$ is a given constant.

4. Integration of the basic equation. Let us change variables by setting $D = Y^{-1}$ in nonlinear matrix equation (3.2). Clearly (see also [4]), this equation then becomes

$$dY/dt = -A'Y - YA + V - YKY, \qquad Y(t_0) = D_0^{-1} \tag{4.1}$$

Let us note the cases where Eq. (3.2) or the equivalent Eq. (4.1) is integrable either exactly or approximately. We denote by $X(t)$ the fundamental matrix of solutions of the following linear homogeneous system with the matrix A :

$$dX / dt = AX, \quad X(t_0) = E \tag{4.2}$$

1°. Let there be no measurements and let the perturbations acting on the system be equal to zero, i.e. let $V = K \equiv 0$. The general solution of systems (3.2) and (4.1), which are linear and homogeneous in this case, be of the form [4]

$$D(t) = X(t) C X'(t), \quad Y(t) = [X'(t)]^{-1} C^{-1} X^{-1}(t) \tag{4.3}$$

This can be verified directly using Eq. (4.2). Determining the constant matrix C with allowance for initial conditions (3.2), (4.1), we obtain $C = D_0$.

2°. If $V(t) \equiv 0$ (i.e. if there are no measurements) and if $K(t)$ is arbitrary, then the solution of linear inhomogeneous system (3.2) can be obtained by applying the method of variation of arbitrary constants and making use of solution (4.3) of the corresponding homogeneous system. Carrying out the usual computations and satisfying initial condition (3.2), we obtain

$$D(t) = X(t) \left\{ D_0 + \int_{t_0}^{t} X^{-1}(\tau) K(\tau) [X'(\tau)]^{-1} d\tau \right\} X'(t) \tag{4.4}$$

3°. For $K(t) \equiv 0$ (zero perturbations acting on the object) system (3.2) remains nonlinear, but system (4.1) is linear and inhomogeneous. It is integrable by the method of variation of arbitrary constants using solution (4.3) of the corresponding homogeneous system. This yields

$$Y(t) = D^{-1}(t) = [X'(t)]^{-1} \left[D_0^{-1} + \int_{t_0}^{t} X'(\tau) V(\tau) X(\tau) d\tau \right] X^{-1}(t) \tag{4.5}$$

which is analogous to (4.4).

Solution (4.5) is obtained in [4], where optimal tracking problems for $K \equiv 0$ are also considered.

4°. Let the tracking errors be large, i.e. let the matrix V be expressible in the form $V = \varepsilon V_*$, where V_* is a matrix with bounded elements and $\varepsilon \ll 1$ is a small parameter. This enables us to seek the solution of system (3.2) in the form $D = D^\circ + \varepsilon D^1$, where D° is given by Eq. (4.4). The matrix D^1 satisfies the equation and initial condition

$$dD^1 / dt = AD^1 + D^1 A' - D^\circ V_* D^\circ, \quad D^1(t_0) = 0 \tag{4.6}$$

to within higher-order small terms.

The solution of linear inhomogeneous system (4.6) can be constructed in the same way as (4.4). It turns out to be

$$D^1(t) = - X(t) \left\{ \int_{t_0}^{t} X^{-1}(\tau) D^\circ(\tau) V_*(\tau) D^\circ(\tau) [X'(\tau)]^{-1} d\tau \right\} X'(t) \tag{4.7}$$

We have therefore constructed an approximate solution of Cauchy problem (3.2).

5°. Let the intensity of the external perturbations be small, i.e. let $K = \varepsilon K_*$, where K_* is a matrix with bounded elements, $\varepsilon \ll 1$. We shall seek the solution of problem (4.1) in the form $Y = Y^\circ + \varepsilon Y^1$, where Y° is given by Eq. (4.5). For Y^1 we construct a linear inhomogeneous problem similar to problem (4.6). Its solution is similar in form to Eq. (4.7),

$$Y^1(t) = - [X'(t)]^{-1} \left[\int_{t_0}^{t} X'(\tau) Y^\circ(\tau) K_*(\tau) Y^\circ(\tau) X(\tau) d\tau \right] X^{-1}(t) \tag{4.8}$$

We have thus obtained an approximate solution $D = Y^{-1} = (Y^\circ + \varepsilon Y^1)^{-1}$ of problem (3.2).

6°. Let us consider the important case where the measurements are made at discrete instants, i.e. where

$$V(t)=\sum_{k=1}^{r} V_k(t)\,\delta(t-t_k), \qquad t_0\leqslant t_1\leqslant\ldots\leqslant t_r\leqslant T \tag{4.9}$$

Here t_k are the instants of measurement, r is their number, δ is a delta function, and $V_k(t)$ are matrix functions characterizing the composition and errors of individual measurements. In the intervals between measurements, i.e. for $t_k<t<t_{k+1}$ the solution of system (3.2) is similar in form to (4.4),

$$D(t)=X(t)\left\{C_k+\int_{t_k}^{t} X^{-1}(\tau)\,K(\tau)\,[X'(\tau)]^{-1}\,d\tau\right\}X'(t) \tag{4.10}$$

$$C_0=D_0,\ t_k<t<t_{k+1}$$

Here C_k are constant matrices. Formulas (1.3) and notation (2.2), (3.1) give us the equations

$$D^{-1}(t_k+0)=D^{-1}(t_k-0)+V_k(t_k),\quad k=1,\ldots,r \tag{4.11}$$

The relations at discontinuities (4.11) give us the relationships between the constants C_k for neighboring intervals. Equations (4.10), (4.11) represent the solution of problem (3.2) in case (4.9). These relations enable us to solve the problem of optimal choice of the measurement instants t_k in the interval $[t_0, T]$ from the standpoint of minimization of one of the functionals of the form (3.4). Here we can impose various additional restrictions on the other functionals of the type (3.4), on the choice of the instants of measurement or their number, etc. Using relations (4.10), (4.11), we can reduce all of these problems to linear programing problems (on the minimization of functions of the variables t_k).

5. The problem with continuous tracking. Let equation of motion (2.3) be of the form

$$dx(t)=[ax(t)+b(t)+\xi(t)]\,dt \tag{5.1}$$

Here $x(t)$ is the sole phase coordinate, a is a constant, $b(t)$ is a given function, and $\xi(t)$ is the scalar random perturbation with the constant dispersion g. The tracking process consists in measuring the present value of the phase coordinate; the dispersion of the measurement error per unit time at any instant is equal to the constant B_0 or to infinity (when no measurements are made). In the notation adopted in Sects. 2–4 we have

$$n=m=l=1,\quad F=Q=1,\quad A=a,\quad V=0$$

or

$$B_0^{-1},\quad K=G=g$$

where all of the matrices and vectors become scalars.

Equation (4.1), which is equivalent to Eq. (3.2), becomes

$$dY/dt=-2aY-qY^2+V,\quad Y=D^{-1} \tag{5.2}$$

Let the sum duration of the measurements be given and equal to $T_0<T$, let the initial condition be of the form $D(0)=D_0$, and let our task be to find a tracking procedure which minimizes the dispersion $D(T)$ of the phase coordinate $x(T)$ at the end of the process. The initial conditions, restrictions, and functional can therefore be written as

$$Y(0)=D_0^{-1},\quad V(t)=0\quad\text{or}\quad V(t)=B_0^{-1},\ J=Y(T)\to\max$$

$$\int_0^T f(V)\,dt=T_0<T,\qquad f(0)=0,\qquad f(B_0^{-1})=1 \tag{5.3}$$

We can solve problem (5.2), (5.3) by means of the maximum principle. Let us construct the Hamiltonian, the associated equation, and the transversality condition

$$H = p\,(-2aY - gY^2 + V) + p_0\, f\,(V) \to \max \text{ with respect to } V$$
$$dp\,/\,dt = 2p\,(a + gY), \qquad p\,(T) = 1 \tag{5.4}$$

Here $p\,(t)$ is the associated variable and p_0 is a constant. Maximum principle (5.4) and restrictions (5.3) on V imply that V can be determined from the condition

$$V\,(t) = B_0^{-1} \quad \text{for} \quad \varphi\,(t) > 0, \qquad V\,(t) = 0 \quad \text{for} \quad \varphi\,(t) < 0$$
$$\varphi\,(t) = p_0 + p\,(t)\,B_0^{-1} \tag{5.5}$$

It is clear that none of the solutions of associated equation except the trivial solution $p\,(t) \equiv 0$ vanish anywhere. Since $p\,(T) > 0$, this fact implies that $p\,(t) > 0$ for $0 \leqslant t \leqslant T$. Making use of Eqs. (5.2) and (5.4), we obtain

$$d^2p\,/\,dt^2 = 2dp\,/\,dt\,(a + gY) + 2pg\,dY\,/\,dt = 2p\,[2(a + gY)^2 +$$
$$+ g\,(-2aY - gY^2 + V)] = 2p\,[(a + gY)^2 + a^2 + gV] > 0$$

The latter inequality follows from the fact that $V \geqslant 0$, $p > 0$. For the function $\varphi\,(t)$ defined by Eq. (5.5) we have, as for $p\,(t)$, the inequality $d^2\varphi\,/\,dt^2 > 0$, i.e. $\varphi\,(t)$ is a strictly convex function having not more than two tracking segments situated at the beginning and end of the interval (one of these segments may be lacking), i.e.

$$\begin{aligned} &V\,(t) = B_0^{-1}, \quad 0 < t < t_* \;\; (0 \leqslant t_* \leqslant T_0) \\ &V\,(t) = 0, \quad t_* < t < T - (T_0 - t_*) \\ &V\,(t) = B_0^{-1}, \quad T - (T_0 - t_*) < t < T \end{aligned} \tag{5.6}$$

Here t_* is the temporarily unknown duration of the first tracking interval; the duration of the second interval is clearly $T_0 - t_*$. If $V\,(t)$ is constant, then Eq. (5.2) is immediately integrable. Its general solution in segments (5.6) is of the form

$$Y\,(t) = \frac{\beta - a}{g} - \frac{2\beta}{g\,(C_1 e^{2\beta t} + 1)}, \quad \beta = \sqrt{a^2 + gB_0^{-1}}, \quad 0 < t < t_*$$

$$Y\,(t) = -\frac{2a}{g\,[C_2\, e^{2a(t - t_*)} + 1]}, \quad t_* < t < T - T_0 + t_* \tag{5.7}$$

$$Y\,(t) = \frac{\beta - a}{g} - \frac{2\beta}{g\,[C_3\, e^{2\beta(t - T + T_0 - t_*)} + 1]}, \qquad T - T_0 + t_* < t < T$$

The arbitrary constants C_1, C_2, C_3 can be determined from initial condition (5.3) and from the conditions of continuity of the function $Y\,(t)$ at the switching points. These conditions yield

$$\frac{\beta - a}{g} - \frac{2\beta}{g\,(C_1 + 1)} = D_0^{-1}, \quad \beta - a - \frac{2\beta}{C_1\, e^{2\beta t_*} + 1} = -\frac{2a}{C_2 + 1}$$

$$-\frac{2a}{C_2 e^{2a(T - T_0)} + 1} = \beta - a - \frac{2\beta}{C_3 + 1}$$

From this we obtain the required constants; we can then use Eqs. (5.3), (5.7) to find the value of the functional $J\,(\lambda)$,

$$C_1 = \frac{\beta + a + gD_0^{-1}}{\beta - a - gD_0^{-1}}, \qquad C_2 = \frac{\gamma C_1 \lambda - 1}{\gamma - C_1 \lambda}$$

$$C_3 = \frac{\gamma C_2 \lambda_1^2 + 1}{\gamma + C_2 \lambda_1^2} = \frac{\gamma C_1 (\gamma^2_1 - 1) - \gamma\,(\lambda_1^2 - 1)}{\gamma \lambda C_1 (\lambda_1^2 - 1) + \gamma^2 - \lambda_1^2} \tag{5.8}$$

$$J\,(\lambda) = \frac{\beta - a}{g} - \frac{2\beta}{g\,(C_3\, e^{2\beta T_0} \lambda^{-1} + 1)}, \; \lambda = e^{2\beta t_*}, \quad \lambda_1 = e^{a(T - T_0)}, \quad \gamma = \frac{\beta + a}{\beta - a} > 0$$

The quantities introduced above satisfy the following inequalities:

$$\gamma \geqslant 1,\quad \lambda_1 \geqslant 1,\quad C_1 \geqslant 1 \quad \text{for} \quad a \geqslant 0, \qquad \gamma \leqslant 1,\quad \lambda_1 \leqslant 1 \quad \text{for } a \leqslant 0 \tag{5.9}$$
$$\frac{\lambda_1+\gamma}{\lambda_1+1} \leqslant 1, \qquad \left|\frac{\lambda_1-\gamma}{\lambda_1\gamma-1}\right| \leqslant 1 \qquad \text{for all} \quad a$$

Relations (5.8) enable us to express the functional $J(\lambda)$ in terms of the unknown λ which depends on t_*. The problem therefore reduces to one of finding the maximum of the function $J(\lambda)$ with respect to λ in the interval

$$1 \leqslant \lambda \leqslant e^{2\beta T_0}, \quad 0 \leqslant t_* \leqslant T_0 \tag{5.10}$$

Eliminating C_2 and C_3 from (5.8), we obtain

$$J(\lambda) = \frac{\beta - a}{g} - \frac{2\beta}{g}\,\frac{\lambda^2\gamma C_1(\lambda_1^2-1)+\lambda(\gamma^2-\lambda_1^2)}{\Lambda(\lambda)} \tag{5.11}$$

$$\frac{dJ(\lambda)}{d\lambda} = -\frac{2\beta}{g}\,\frac{e^{2\beta T_0}\gamma(\lambda_1^2-1)\,[\lambda^2C_1^2(\gamma^2\lambda_1^2-1)-2\lambda C_1\gamma(\lambda_1^2-1)-(\gamma^2-\lambda_1^2)]}{\Lambda^2(\lambda)}$$

$$\Lambda(\lambda) = \lambda^2\gamma C_1(\lambda_1^2-1)+\lambda\,[\gamma^2-\lambda_1^2+C_1e^{2\beta T_0}(\gamma^2\lambda_1^2-1)]-\gamma e^{2\beta T_0}(\lambda_1^2-1)$$

Let us denote the values of λ for which $J(\lambda)$ attains a local maximum or minimum by $\lambda_{\max}$ and $\lambda_{\min}$, respectively.

Equating $dJ/d\lambda$ to zero and investigating the sign of $dJ/d\lambda$, we find from (5.11) with allowance for (5.9) that

$$\lambda_{\max} = \frac{1}{C_1}\,\frac{\lambda_1+\gamma}{\lambda_1\gamma+1},\quad \lambda_{\min} = \frac{1}{C_1}\,\frac{\lambda_1-\gamma}{\lambda_1\gamma-1} \quad \text{for } C_1>0$$
$$\lambda_{\max} = \frac{1}{C_1}\,\frac{\lambda_1-\gamma}{\lambda_1\gamma-1},\quad \lambda_{\min} = \frac{1}{C_1}\,\frac{\lambda_1+\gamma}{\lambda_1\gamma+1} \quad \text{for } C_1<0$$
$$\lambda_{\max} > \lambda_{\min} \tag{5.12}$$
$$dJ/d\lambda < 0 \quad \text{for} \quad \lambda > \lambda_{\max},\quad \lambda<\lambda_{\min}$$
$$dJ/d\lambda > 0 \quad \text{for} \quad \lambda_{\min}<\lambda<\lambda_{\max}$$

Let us find the optimal tracking law.

Making use of relations (5.9), (5.12), we find that $\lambda_{\max} \leqslant 1$ for $a \geqslant 0$. Conditions (5.12) indicate that $dJ/d\lambda < 0$ in interval (5.10), i.e. that $J(\lambda)$ decreases in this interval. Hence, $J(\lambda)$ attains its maximum value for $\lambda = 1$. The optimal tracking law for $a \geqslant 0$ is of the form

$$V(t) = 0, \quad 0<t<T-T_0, \qquad V(t) = B_0^{-1}, \qquad T-T_0<t<T \tag{5.13}$$

For $a < 0$ the optimal tracking law is of the form (5.6), where, depending on the parameters of the problem, t_* assumes one of the values

$$\begin{aligned}
t_* &= 0, && \text{if} \quad \lambda_{\max} \leqslant 1 \\
t_* &= 0, && \text{if} \quad 1<\lambda_{\max}\leqslant e^{2\beta T_0},\ J(1)\geqslant J(\lambda_{\max}) \\
t_* &= \ln(\lambda_{\max})/2\beta, && \text{if} \quad 1<\lambda_{\max}\leqslant e^{2\beta T_0},\ J(1)<J(\lambda_{\max}) \\
t_* &= T_0, && \text{if} \quad \lambda_{\min}\leqslant 1,\ \lambda_{\max}>e^{2\beta T_0} \\
t_* &= 0, && \text{if} \quad \lambda_{\min}>1,\ \lambda_{\max}>e^{2\beta T_0},\ J(1)\geqslant J(e^{2\beta T_0}) \\
t_* &= T_0, && \text{if} \quad \lambda_{\min}>1,\ \lambda_{\max}>e^{2\beta T_0},\ J(1)<J(e^{2\beta T_0})
\end{aligned} \tag{5.14}$$

Relations (5.14) are obtainable from the condition of maximality of $J(\lambda)$ in interval (5.10) with allowance for conditions (5.12).

Thus, Eqs. (5.13) and (5.5), (5.14) constitute the optimal tracking law for $a \geqslant 0$ and

$a < 0$, respectively.

Comparing these results with those obtained in [4] for a similar example in the absence of external perturbations ($g = 0$), we find that these perturbations generally have the effect of shifting the measurements towards the end of the interval $[0, T]$.

6. Discrete-tracking problems. 1°. Let the equation of motion of the system be of the form

$$dx(t) = [b(t) + \xi(t)]\,dt \tag{6.1}$$

Here, as in the previous problem, $x(t)$ is the sole phase coordinate, $\xi(t)$ is the scalar random perturbation with the constant dispersion g, and $b(t)$ is a given function. Let the measurements of the phase coordinate be made at the discrete instants t_k, $k = 1, \ldots, r$, i.e. let

$$V(t) = \sum_{k=1}^{r} V_k \delta(t - t_k), \qquad t_0 \leqslant t_1 \leqslant \ldots \leqslant t_r \leqslant T$$

where the V_k are constant.

We are required to choose the tracking instants in such a way as to minimize the dispersion $D(T)$ of the phase coordinate at the end of the process.

Equation (4.1) for this problem becomes

$$dY / dt = -gY^2 + V, \qquad Y(t_0) = D_0^{-1} \tag{6.2}$$

Using formula (4.11) we find that

$$Y(t_k + 0) = Y(t_k - 0) + V_k \qquad (k = 1, \ldots, r) \tag{6.3}$$

Integrating Eq. (6.2) over the interval (t_{k-1}, t_k) with allowance for Eqs. (6.3), we obtain

$$Y(t_k + 0) = \frac{Y(t_{k-1} + 0)}{1 + (t_k - t_{k-1})gY(t_{k-1} + 0)} + V_k \qquad (k = 1, \ldots, r) \tag{6.4}$$

$$D^{-1}(T) = \frac{Y(t_r + 0)}{1 + (T - t_r)gY(t_r + 0)}$$

Let the instants t_{r-1} and the value of $Y(t_{r-1} + 0)$ be known. Let us find the t_r such that $t_{r-1} \leqslant t_r \leqslant T$ and $D^{-1}(T)$ are maximum. Making use of (6.4), we obtain

$$D^{-1}(T) = \frac{V_r + Y(t_{r-1} + 0)[1 + (t_r - t_{r-1})gV_r]}{1 + (T - t_{r-1})gY(t_{r-1} + 0) + (T - t_r)gV_r[1 + (t_r - t_{r-1})gY(t_{r-1} + 0)]}$$

It is clear that $D^{-1}(T)$ attains its maximum value at $t_r = T$. Reasoning by induction as above, we obtain the following optimal tracking law:

$$t_1 = \ldots = t_r = T$$

i.e. all of the measurements are made at the end of the process.

2°. Let us take the equations of motion of the system in the form

$$dx_1(t) / dt = x_2, \qquad dx_2(t) = [b(t) + \xi(t)]\,dt \tag{6.5}$$

Here $x_1(t)$ is the coordinate, $x_2(t)$ is the velocity, $b(t)$ is a given function, and $\xi(t)$ is the random perturbation. We assume that the intensity of the perturbation is low, i.e. that

$$K(t) = \varepsilon K_*(t) = \varepsilon \begin{Vmatrix} 0 & 0 \\ 0 & g_* \end{Vmatrix} \tag{6.6}$$

Here $\varepsilon \ll 1$ is a small parameter and g_* is a constant.

Let measurements of the coordinate be made at the discrete instants t_k, $k = 1, \ldots, r$, and let the measurement errors be large,

$$V(t)=\varepsilon\sum_{k=1}^{r}V_k\,\delta(t-t_k)\begin{Vmatrix}1&0\\0&0\end{Vmatrix},\qquad t_0\leqslant t^1\leqslant\ldots\leqslant t_r\leqslant t_{r+1}=T$$

Here V_k are constants. We are required to choose the instants of measurement in such a way as to minimize the dispersion of the coordinate at the end of the process.

Making use of relations (4.10), (4.11) and limiting ourselves to terms of the first order of smallness in ε, we obtain the following recursion relations:

$$D(t_k+0)=D(t_{k-1}+0)+$$
$$+X(t_k)\left\{\int_{t_{k-1}}^{t_k}X^{-1}(\tau)K(\tau)[X'(\tau)]^{-1}d\tau\right\}X'(t_k)-\varepsilon D(t_{k-1}+0)V_k\begin{Vmatrix}1&0\\0&0\end{Vmatrix}D(t_{k-1}+0),$$
$$D(t_0)=D_0\qquad(k=1,\ldots,r)\tag{6.7}$$

Here $X(t)$ is the fundamental matrix of solutions of the linear homogeneous system. it is given by

$$X(t)=\begin{Vmatrix}1&t\\0&1\end{Vmatrix}\tag{6.8}$$

Recursion relations (6.7) together with (6.6), (6.8) yield the following equations accurate to within higher-order terms:

$$D(T)=D_0+\varepsilon g_*\begin{Vmatrix}d_{11}&d_{12}\\d_{21}&T\end{Vmatrix}-\varepsilon\sum_{k=1}^{r}V_kD_0\begin{Vmatrix}1&0\\0&0\end{Vmatrix}D_0$$
$$d_{11}=\frac{1}{3}\sum_{k=1}^{r+1}(t_k-t_{k-1})^3,\qquad d_{12}=d_{21}=\frac{1}{2}\sum_{k=1}^{r+1}(t_k-t_{k-1})^2\tag{6.9}$$

The problem of minimum dispersion of the coordinate reduces to finding the minimum of the element

$$d_{11}=\frac{1}{3}\sum_{k=1}^{r+1}(t_k-t_{k-1})^3,\qquad t_0\leqslant t_1\leqslant\ldots\leqslant t_r\leqslant t_{r+1}=T$$

It is easy to show that the required minimum is attained for

$$t_k=\frac{T-t_0}{r+1}k\qquad(k=1,\ldots,r)$$

Thus, the optimal tracking law for the above problem is as follows: all of the instants of measurement must be distributed uniformly over the interval $[t_0,T]$.

We note that the dispersion of the velocity at the end of the process does not depend on the choice of instants of measurement.

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